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The stress state of an elastic composite cone with centre of rotation at the apex☆

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Abstract

The torsion of a composite cone that has a centre of rotation at its apex is investigated in a spherical system of coordinates. A composite cone is a cone with one shear modulus, inserted into a conical funnel having another shear modulus and with ideal mechanical contact between its surface and the inner surface of the conical funnel. The auxiliary problem of a composite cone with its apex truncated by a spherical surface is considered first. The outer surface of such a conical body is not loaded, but a load that reduces to a torque is applied to its spherical surface. The auxiliary problem is reduced to a one-dimensional discontinuous boundary-value problem using a specially constructed integral transformation. The exact solution of this boundary-value problem is constructed. The limit is then taken in the solution obtained as the radius of the spherical surface tends to zero for the purpose of obtaining an exact solution of the problem of the torsion of a composite cone that has a centre of rotation at the apex.

It was shown in Ref. 1 for the case of a homogeneous cone that the presence of a centre of rotation at the apex does not cause a stress concentration on internal cuts along conical surfaces. According to the results obtained below, this is also true for a composite cone, but not if the cone's apex is removed and shear stresses equivalent to a torque are applied to a spherical cut of the surface.

1. Formulation of the problem

A composite cone in a spherical system of coordinates r, θ, φ occupies the region $0 \le r < \infty$, $0 \le \theta \le \omega$, $-\pi \le \varphi < \pi$, and the value of the shear modulus G changes abruptly on the internal conical surface $\theta = \omega_0 < \omega$, i.e.,

$$G = \begin{cases} G_0, & 0 \le \theta \le \omega_0 \\ G_1, & \omega_0 < \theta \le \omega \end{cases} \tag{1.1}$$

In other words, this surface is a defect,[2] i.e., the derivative of the displacement sought $u_\varphi(r, \theta)$ with respect to a normal to this surface undergoes a discontinuity of the first kind on passing through it. We assume that the following matching

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conditions hold at the defect points $\theta = \omega_0$

$$u_\varphi(r, \omega_0 - 0) = u_\varphi(r, \omega_0 + 0), \quad \tau_{\theta\varphi}(r, \varphi_0 - 0) = \tau_{\theta\varphi}(r, \omega_0 + 0) \tag{1.2}$$

As was done previously in Ref. 1, we first consider the auxiliary problem of a similar cone with a truncated apex. This problem is formulated as follows. It is required to solve the torsion equation[3]

$$[r^2 u'_\varphi(r, \theta)]' + \frac{[\sin\theta \dot{u}_\varphi(r, \theta)]^{\bullet}}{\sin\theta} + \frac{u_\varphi(r, \theta)}{\sin^2\theta} = 0, \quad a < r < \infty, \quad 0 < \theta < \omega \tag{1.3}$$

where the prime denotes a derivative with respect to the first variable, and the dot denotes a derivative with respect to the second variable. Eq. (1.3) must be satisfied everywhere in the region demarcated in (1.3), apart from the points $\theta = \omega_0$. At these points, the matching conditions (1.2) should hold. When the formulae[3]

$$\tau_{\theta\varphi} = Gr^{-1}[\dot{u}_\varphi(r, \theta) - \operatorname{ctg}\theta u_\varphi(r, \theta)], \quad \tau_{r\varphi} = Gr[r^{-1}u_\varphi(r, \theta)]' \tag{1.4}$$

are taken into account, it is convenient to write these conditions in the form

$$\begin{gathered} u_\varphi(r, \omega_0 - 0) - u_\varphi(r, \omega_0 + 0) \equiv \langle u_\varphi(r, \omega_0)\rangle = 0 \\ \langle \dot{u}_\varphi(r, \omega_0)\rangle = -g_{01}[\dot{u}_\varphi(r, \omega_0 - 0) - \operatorname{ctg}\omega_0 u_\varphi(r, \omega_0 - 0)] \\ g_{01} = (G_0 - G_1)G_1^{-1} = g - 1, \quad g = G_0 G_1^{-1} \end{gathered} \tag{1.5}$$

We assume that the outer conical surface of the cone $\theta = \omega$ is not loaded, i.e., $\tau_{\theta\varphi}(r, \omega) = 0$, or, taking formulae (1.4) into account,

$$\dot{u}_\varphi(r, \omega) - \operatorname{ctg}\omega u_\varphi(r, \omega) = 0, \quad a < r < \infty \tag{1.6}$$

We apply a torsional load to the spherical part of the boundary $r = a$ of the truncated cone under consideration, i.e.,

$$\tau_{r\varphi}(a, \theta) = A\sin\theta, \quad A = \text{const} \tag{1.7}$$

which reduces to the torque

$$M = 4\pi A_\omega A a^3, \quad A_\omega = \sin^2\frac{\omega}{2}\left[1 - \frac{1}{3}(\cos^2\omega + \cos\omega + 1)\right] \tag{1.8}$$

We rewrite the boundary condition (1.7) taking relations (1.1) and (1.4) into account in the form

$$u'_\varphi(a, 0) - \frac{u_\varphi(a, \theta)}{a} = A\sin\theta\eta(\theta), \quad \eta(\theta) = \begin{cases} G_0^{-1}, & 0 \le \theta \le \omega_0 \\ G_1^{-1}, & \omega_0 < \theta \le \omega \end{cases} \tag{1.9}$$

The discontinuous boundary-value problem defined by (1.3), (1.5), (1.6) and (1.9) is an auxiliary problem, which is of interest in itself. After it is solved, we must take the limit as $a \to 0$ in the solution obtained to solve the main problem. However, the constant A in the first equality in (1.8) must be increased so that

$$Aa^3 = \frac{M}{4\pi A_\omega} \tag{1.10}$$

where M is the moment of the centre of rotation.

We transform the auxiliary problem formulated using the replacement

$$u_\varphi(r, \theta) = r^{-1/2}u(\ln r, \theta) \tag{1.11}$$

and then we set $\ln r = x$. Here the variable x can vary in the range $\ln a < x < \infty$. We will change from the variable x to the variable $\xi = x - \ln a$, which will vary in the range $0 < \xi < \infty$. By virtue of these replacements, we have the equalities

$$u(\ln r, \theta) = u(x, \theta) = u(\xi + \ln a, \theta) = \tilde{u}(\xi, \theta) \tag{1.12}$$

We take the function $\tilde{u}(\xi, \theta)$ as the function required. If it is found, then according to the relations (1.11) and (1.12), we can write

$$u_\varphi(r, \theta) = r^{-1/2}\tilde{u}(\ln r/a, \theta) \tag{1.13}$$

The auxiliary problem (1.3), (1.5), (1.6), (1.9) was formulated for the function introduced $\tilde{u}(\xi, \theta)$, taking (1.11) and (1.12) into account in the form of the following two-dimensional discontinuous boundary-value problem ($\theta \neq \omega_0$)

$$\begin{gathered}\tilde{u}''(\xi,\theta) - \frac{1}{4}\tilde{u}(\xi,\theta) + \frac{[\sin\theta\tilde{u}^{\bullet}(\xi,\theta)]^{\bullet}}{\sin\theta} - \frac{\tilde{u}(\xi,\theta)}{\sin^2\theta} = 0, \quad 0<\xi<\infty, \quad 0<\theta<\omega \\ \tilde{u}'(0,\theta) - \operatorname{ctg}\alpha\tilde{u}(0,\theta) = Aa^{3/2}\eta(\theta)\sin\theta, \quad \operatorname{ctg}\alpha = 3/2 \\ \tilde{u}^{\bullet}(\xi,\omega) - \operatorname{ctg}\omega\tilde{u}(\xi,\omega) = 0, \quad 0\le\xi<\infty \\ \langle\tilde{u}(\xi,\omega_0)\rangle = 0, \quad \langle\tilde{u}^{\bullet}(\xi,\omega_0)\rangle = -g_{01}[\tilde{u}^{\bullet}(\xi,\omega_0-0) - \operatorname{ctg}\omega_0\tilde{u}(\xi,\omega_0-0)]; \quad 0\le\xi<\infty\end{gathered} \tag{1.14}$$

2. Reduction of the auxiliary problem to a one-dimensional discontinuous boundary-value problem

To reduce boundary-value problem (1.14) to a one-dimensional problem, an integral transformation with respect to the variable ξ must be used so that the boundary condition at $\xi = 0$ is satisfied.

Let us obtain such a transformation. For this purpose, we solve the following singular Sturm–Liouville equation[4–6]

$$-y''(x,\lambda) = \lambda y(x,\lambda), \quad 0<x<\infty; \quad \cos\alpha y(0,\lambda) - \sin\alpha y'(0,\lambda) = 0 \tag{2.1}$$

The boundary condition in problem (2.1) is equivalent to the following

$$y'(0,\lambda) - \operatorname{ctg}\alpha y(0,\lambda) = 0$$

According to (1.14), the case of $\cot\alpha = 3/2$ is important for what follows.

We will seek the solution of boundary-value problem (2.1) using a well-known method.[4,5] We construct two linearly independent solutions $\chi(x, \lambda)$ and $\psi(x, \lambda)$ of Eq. (2.1) that satisfy the conditions

$$\chi(0,\lambda) = \cos\alpha, \quad \chi'(0,\lambda) = -\sin\alpha; \quad \psi(0,\lambda) = \cos\alpha, \quad \psi'(0,\lambda) = \sin\alpha$$

They have the form

$$\begin{aligned}\chi(x,\lambda) &= -\lambda^{-1/2}\sin\alpha\sin(x\sqrt{\lambda}) + \cos\alpha\cos(x\sqrt{\lambda}) \\ \psi(x,\lambda) &= \lambda^{-1/2}\cos\alpha\sin(x\sqrt{\lambda}) + \sin\alpha\cos(x\sqrt{\lambda})\end{aligned} \tag{2.2}$$

Besides problem (2.1), we consider the auxiliary boundary-value problem

$$\begin{gathered}-y''(x,\lambda) = \lambda y(x,\lambda), \quad 0<x<b \\ \cos\alpha y(0,\lambda) - \sin\alpha y'(0,\lambda) = 0, \quad \cos\beta y(b,\lambda) - \sin\beta y'(b,\lambda) = 0\end{gathered} \tag{2.3}$$

The general solution of the differential Eq. (2.3) is

$$y(x,\lambda) = \chi(x,\lambda) + m(\lambda, b; \beta)\psi(x,\lambda) \tag{2.4}$$

and we can find an expression for $m(\lambda, b; \beta)$ after satisfying the second boundary condition of problem (2.3). It will simultaneously define the equation of a circle[4,5] in the plane of the complex variable $m(\lambda, b; \beta)$, and all points on the circle will be by-passed as the parameter β varies from zero to π. As b increases, the circle shrinks to a limit point because the integral $\int_0 |\psi(x, \lambda)|^2 dx$ diverges.[4,5] We will use $m_\infty(\lambda)$ to denote the corresponding limit value of $m(\lambda, b; \beta)$ in (2.4). From the condition[4,5] for complex λ to exist, we find at least one function (2.4) belonging to $L^2(0, \infty)$

$$m_\infty(\lambda) = \frac{1 + i\sqrt{\lambda}\operatorname{ctg}\alpha}{\operatorname{ctg}\alpha - i\sqrt{\lambda}}$$

Hence, taking into account that $\operatorname{ctg}\alpha > 0$ in the case under consideration, we have

$$\operatorname{Im} m_\infty(\lambda) = 0, \quad \lambda < 0; \quad \operatorname{Im} m_\infty(\lambda) = \frac{\sqrt{\lambda}}{\sin^2\alpha(\lambda + \operatorname{ctg}\alpha)}, \quad \lambda > 0 \tag{2.5}$$

and we find the eigenvalue function

$$d\sigma(\lambda) = \frac{1}{\pi}\lim_{\varepsilon\to 0}\operatorname{Im} m_\infty(\lambda + i\varepsilon)d\lambda = \frac{\sqrt{\lambda}d\lambda}{\pi\sin^2\alpha(\lambda + \operatorname{ctg}^2\alpha)}$$

Therefore, when relations (2.5) and the last equality are taken into account, the following expansion with convergence on the average[4,5] holds for any function $g(x)$ from $L^2(0, \infty)$

$$g(x) = \frac{1}{\pi}\int_0^\infty \frac{\psi(x,\lambda)\sqrt{\lambda}}{\sin^2\alpha(\lambda + \operatorname{ctg}^2\alpha)}\left(\int_0^\infty \psi(\xi,\lambda)g(\xi)d\xi\right)d\lambda \tag{2.6}$$

If we represent the function defined by the second equality in (2.2) in the form

$$\psi(x,\lambda) = \lambda^{-1/2}\sin\alpha\varphi_\lambda(x), \quad \varphi_\lambda(x) = \operatorname{ctg}\alpha\sin(x\sqrt{\lambda}) + \sqrt{\lambda}\cos(x\sqrt{\lambda}) \tag{2.7}$$

we can regard relation (2.6) as an integral transformation for the function $g(x)$:

$$g_\lambda = \int_0^\infty g(x)\varphi_\lambda(x)dx, \quad g(x) = \frac{1}{\pi}\int_0^\infty \frac{\varphi_\lambda(x)g_\lambda d\lambda}{\sqrt{\lambda}(\lambda + \operatorname{ctg}^2\alpha)} \tag{2.8}$$

The former equality defines the transform, and the latter equality is the inversion formula.

It follows from the derivation that the function $\varphi_\lambda(x)$ satisfies the following differential equation and boundary condition

$$-\varphi_\lambda''(x) = \lambda\varphi_\lambda(x), \quad 0 < x < \infty; \quad \varphi_\lambda'(0) - \operatorname{ctg}\alpha\varphi_\lambda(0) = 0 \tag{2.9}$$

The Sturm–Liouville problem (2.2) can also be investigated by Titchmarsh's method, which was previously described in Ref. 6, where an example similar to (2.1) was considered.

We apply the integral transformation (2.8) to boundary-value problem (1.14). Instead of the function u (ξ, θ) we seek its transform

$$\tilde{u}_\lambda(\theta) = \int_0^\infty \varphi_\lambda(\xi)\tilde{u}(\xi,\theta)d\xi \tag{2.10}$$

If it is found, then, by inversion formula (2.8), we will have

$$\tilde{u}(\xi,\theta) = \frac{1}{\pi}\int_0^\infty \frac{\varphi_\lambda(\xi)\tilde{u}_\lambda(\theta)d\lambda}{\sqrt{\lambda}(\lambda + \operatorname{ctg}^2\alpha)} \tag{2.11}$$

Taking relations (2.9) into account, we obtain the following one-dimensional boundary-value problem for the transform (2.10) from problem (1.14)

$$\begin{gathered}
L_s\tilde{u}_\lambda(\theta) \equiv -[\sin\theta\dot{\tilde{u}}_\lambda(\theta)]^{\cdot} + \frac{\tilde{u}_\lambda(\theta)}{\sin\theta} + \left(\lambda + \frac{1}{4}\right)\sin\theta\tilde{u}_\lambda(\theta) = g(\theta) \\
g(\theta) = -Aa^{3/2}\sqrt{\lambda}h(\theta), \quad h(\theta) = \eta(\theta)\sin^2\theta, \quad 0 < \theta < \omega, \quad \theta \neq \omega_0 \\
\tilde{u}_\lambda(\omega) - \operatorname{ctg}\omega\tilde{u}_\lambda(\omega) = 0, \quad \langle\tilde{u}_\lambda(\omega_0)\rangle = 0, \quad \langle\dot{\tilde{u}}_\lambda(\omega_0)\rangle = -g_{01}X_\lambda(\omega_0)
\end{gathered} \tag{2.12}$$

where

$$X_\lambda(\omega_0) = \tilde{u}^{\bullet}_\lambda(\omega_0 - 0) - \operatorname{ctg}\omega_0 \tilde{u}_\lambda(\omega_0 - 0) \tag{2.13}$$

According to the theory previously described,[2] to solve the one-dimensional discontinuous boundary-value problem (2.12), we must construct Green's function $G_\lambda(\theta, t)$ of the boundary-value problem

$$L_s y(\theta) = f(\theta), \quad 0 < \theta < \omega, \quad y^{\bullet}(\omega) - \operatorname{ctg}\omega y(\omega) = 0 \tag{2.14}$$

3. The construction of Green's function of boundary-value problem (2.14)

We shall construct Green's function of boundary-value problem (2.14) from its defining properties.[2] The boundary-value problem is self-conjugate;[2] therefore, Green's function should be symmetric, i.e.,

$$G_\lambda(\theta, t) = G_\lambda(t, \theta) \tag{3.1}$$

It can be shown that the fundamental system of solutions of the homogeneous differential equation from (2.14) will be the spherical functions $P^1_{\bar{\nu}}(\cos\theta)$ and $Q^1_{\bar{\nu}}(\cos\theta)$ for $\bar{\nu} = -1/2 - ip$, $p = \sqrt{\lambda}$, the former function being real[7] and the latter being complex. We will construct the general real solution of the equation in the form

$$y(\theta) = C_0 P^1_{\bar{\nu}}(\cos\theta) + C_1 \operatorname{Re} Q^1_{\bar{\nu}}(\cos\theta) \tag{3.2}$$

Using this solution, we construct the function

$$\psi_p(\theta) = P^1_{\bar{\nu}}(\cos\theta) - C_p(\omega)\operatorname{Re} Q^1_{\bar{\nu}}(\cos\theta) \tag{3.3}$$

where

$$C_p(\omega) = \frac{P^{1\bullet}_{\bar{\nu}}(\cos\omega) - \operatorname{ctg}\omega P^1_{\bar{\nu}}(\cos\omega)}{\operatorname{Re} Q^{1\bullet}_{\bar{\nu}}(\cos\omega) - \operatorname{ctg}\omega \operatorname{Re} Q^1_{\bar{\nu}}(\cos\omega)} = \frac{P^2_{\bar{\nu}}(\cos\omega)}{\operatorname{Re} Q^2_{\bar{\nu}}(\cos\omega)} \tag{3.4}$$

The latter equality follows from well-known formulae (Ref. 7, formulae 3.6.1(6) and 3.6.1(7)). Function (3.3) satisfies the homogeneous differential equation and the boundary condition of problem (2.12).

In order for condition (3.1) and the boundary condition of problem (2.12) to be satisfied for Green's function $G_\lambda(\theta, t)$, it must have the form

$$G_\lambda(\theta, t) = c_0 \begin{cases} P^1_{\bar{\nu}}(\cos\theta)\psi_p(t), & \theta < t, \\ \psi_p(\theta) P^1_{\bar{\nu}}(\cos t), & \theta > t, \end{cases} \quad \bar{\nu} = -\frac{1}{2} - ip, \quad p = \sqrt{\lambda} \tag{3.5}$$

We find the constant c_0 from the discontinuity condition of the first derivative of Green's function with respect to θ.[2] As a result, we obtain

$$c_0 = [C_p(\omega)(\lambda + 1/4)]^{-1} \tag{3.6}$$

Here we have used a well-known formula (Ref. 7, formula 3.4.(25)) that enables us to calculate the Wronskian of the functions on the right-hand side of equality (3.2), i.e.,

$$P^1_{\bar{\nu}}(\cos\theta)\operatorname{Re} Q^{1\bullet}_{\bar{\nu}}(\cos\theta) - P^{1\bullet}_{\bar{\nu}}(\cos\theta)\operatorname{Re} Q^1_{\bar{\nu}}(\cos\theta) = (\lambda + 1/4)(\sin\theta)^{-1} \tag{3.7}$$

as well as the functions $P^1_{\bar{\nu}}(\cos\theta)$ and $\psi_p(\theta)$.

4. Solution of the one-dimensional discontinuous boundary-value problem (2.12)

According to the theory previously presented in Ref. 2, the solution of the discontinuous boundary-value problem (2.12) has the form

$$\tilde{u}_\lambda(\theta) = -a^{3/2}A\sqrt{\lambda}\int_0^\omega G_\lambda(\theta, t)h(t)dt - g_{01}\sin\omega_0 X_\lambda(\omega_0)G_\lambda(\theta, \omega_0) \tag{4.1}$$

Taking into account the discontinuity of $h(t)$, as well as the representation of Green's function (3.5), (3.6), we can write solution (4.1) in the form

$$\begin{aligned}\tilde{u}_\lambda(\theta) = &-\frac{1}{(\lambda+1/4)P_{\bar{\nu}}^2(\cos\omega)}\left[a^{3/2}A\sqrt{\lambda}\left(\frac{R^p_{0,\theta}+S^p_{\theta,\omega_0}}{G_0}+\frac{S^p_{\omega_0,\omega}}{G_1}\right)-\right.\\ &\left.-g_{01}\sin\omega_0 X_\lambda(\omega_0)Q_\omega(p,\omega_0,\theta)\right],\quad 0\le\theta\le\omega_0\end{aligned} \tag{4.2}$$

Here we have introduced the notation

$$\begin{aligned}&Q_k(p,\theta,\omega) = P^k_{\bar{\nu}}(\cos\theta)\mathrm{Re}Q^2_{\bar{\nu}}(\cos\omega) - \mathrm{Re}Q^k_{\bar{\nu}}(\cos\theta)P^2_{\bar{\nu}}(\cos\omega),\quad k = 1, 2\\ &Q_2(p,\theta,\omega) = Q_1^{\bullet}(p,\theta,\omega),\quad Q_\omega(p,\theta,t) = Q_1(p,\theta,\omega)P^1_\nu(\cos t)\\ &R^p_{\alpha,\beta} = \int_\alpha^\beta Q_\omega(p,\theta,t)\sin^2 tdt,\quad S^p_{\alpha,\beta} = \int_\alpha^\beta Q_\omega(p,t,\theta)\sin^2 tdt\end{aligned} \tag{4.3}$$

and taken into account the equalities

$$\psi_p^{\bullet}(\theta) - \mathrm{ctg}\theta\psi_p(\theta) = \frac{Q_2(p,\theta,\omega)}{P^2_{\bar{\nu}}(\cos\omega)},\quad \frac{\psi_p(\theta)}{C_p(\theta)} = \frac{Q_1(p,\theta,\omega)}{P^2_{\bar{\nu}}(\cos\omega)} \tag{4.4}$$

whose validity can be proved using relations (3.3) and (3.4).

Formula (4.1) can be reduced to a similar form for $\omega_0 \le \theta \le \omega$.

Using formula (4.2), we will evaluate the combination appearing on the right-hand side of (2.13). As a result, we obtain an equation for $X_\lambda(\omega_0)$, from which we find

$$X_\lambda(\omega_0) = -a^{3/2}A\sqrt{\lambda}C_\omega(p,\omega_0)/\sin\omega_0,\quad C_\omega(p,\omega_0) = A_\omega(p,\omega_0)/B_\omega(p,\omega_0) \tag{4.5}$$

where

$$A_\omega(p,\omega_0) = \frac{Q_2(p,\omega_0,\omega)}{G_0}\int_0^{\omega_0}P_{\bar{\nu}}(\cos t)\sin^2 tdt + \frac{P^2_{\bar{\nu}}(\cos\omega_0)}{G_1}\int_{\omega_0}^{\omega}Q_1(p,t,\omega)\sin^2 tdt \tag{4.6}$$

$$\begin{aligned}B_\omega(p,\omega_0) &= (\lambda+1/4)(\sin\omega_0)^{-1}P^2_{\bar{\nu}}(\cos\omega) + g_{01}P^2_{\bar{\nu}}(\cos\omega_0)Q_1(p,\omega_0,\omega) =\\ &= gP^2_{\bar{\nu}}(\cos\omega_0)Q_1(p,\omega_0,\omega) - P^1_{\bar{\nu}}(\cos\omega_0)Q_2(p,\omega_0,\omega)\end{aligned} \tag{4.7}$$

The second equality in (4.7) was obtained using equalities (3.7) and (1.5).

Now the solution of the one-dimensional boundary-value problem (2.12) is given by the formulae

$$\tilde{u}_\lambda(\theta) = -\frac{a^{3/2}A\sqrt{\lambda}}{(\lambda+1/4)P_{\bar{\nu}}^2(\cos\omega)}\left[\frac{R^p_{0,\theta}+S^p_{\theta,\omega_0}}{G_0}+\frac{S^p_{\omega_0,\omega}}{G_1}-g_{01}Q_\omega(p,\omega_0,\theta)C_\omega(p,\omega_0)\right], \quad 0\le\theta\le\omega_0 \tag{4.8}$$

$$\tilde{u}_\lambda(\theta) = -\frac{a^{3/2}A\sqrt{\lambda}}{(\lambda+1/4)P_{\bar{\nu}}^2(\cos\omega)}\left[\frac{R^p_{0,\omega_0}}{G_0}+\frac{R^p_{\omega_0,\omega}+S^p_{\theta,\omega}}{G_1}-g_{01}Q_\omega(p,\theta,\omega_0)C_\omega(p,\omega_0)\right], \quad \omega_0\le\theta\le\omega \tag{4.9}$$

5. Construction of a solution of the two-dimensional discontinuous boundary-value problem (1.14) and an exact solution of the auxiliary problem

We will obtain a solution of boundary-value problem (1.14) using formula (2.11). If we take into account the structure of the function $\varphi_\lambda(\xi)$, that follows from (2.7), and introduce the integrals

$$\begin{Vmatrix} I_0(\xi,\theta) \\ I_1(\xi,\theta) \end{Vmatrix} = \frac{1}{G_0}\begin{Vmatrix} R_{0,\theta}+\tilde{R}_{0,\omega_0} \\ R_{0,\omega_0} \end{Vmatrix} + \frac{1}{G_1}\begin{Vmatrix} \tilde{R}_{\omega_0,\omega} \\ R_{\omega_0,\theta}+\tilde{R}_{\theta,\omega_0} \end{Vmatrix} - g_{01}\begin{Vmatrix} J_1(\xi,\omega_0,\theta) \\ J_1(\xi,\theta,\omega_0) \end{Vmatrix} \tag{5.1}$$

where

$$\begin{Vmatrix} R_{\alpha,\beta}(\xi,\theta) \\ \tilde{R}_{\alpha,\beta}(\xi,\theta) \end{Vmatrix} = \int_\alpha^\beta \sin^2 t \begin{Vmatrix} J_0(\xi,\theta,t) \\ J_0(\xi,t,\theta) \end{Vmatrix} dt \tag{5.2}$$

$$J_n(\xi,\theta,t) = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{2pe^{ip\xi}Q_\omega(p,\theta,t)[\delta_{n0}+\delta_{n1}C_\omega(p,\omega_0)]}{(p^2+1/4)(p^2+9/4)P_{\bar{\nu}}^2(\cos\omega)}dp$$
$$n = 0, 1, \quad \delta_{00}=\delta_{11}=1, \quad \delta_{01}=\delta_{10}=0, \quad \bar{\nu}=-1/2-ip, \quad 0<\xi<\infty \tag{5.3}$$

formula (2.11) can be represented in the form

$$\tilde{u}(\xi,\theta) = -a^{3/2}A\left[\frac{3}{2}I_n(\xi,\theta)+I'_n(\xi,\theta)\right], \quad n = \begin{cases} 0, & 0\le\theta\le\omega_0 \\ 1, & \omega_0<\theta\le\omega \end{cases} \tag{5.4}$$

In writing the integrals (5.3), we made the replacement $\sqrt{\lambda}=p$, $d\lambda=2pdp$ and took into account that the spherical functions in them are even with respect to the variable p.

Expanding formulae (5.4), we can use equality (1.13) to construct a solution of the auxiliary problem, but to do this we must first demonstrate the convergence of the integrals (5.3) and provide a method for evaluating them.

To prove the convergence of these integrals, we must expand the asymptotic formulae for the cone functions $P^m_{-1/2-ip}(\cos\theta)$, $Q^m_{-1/2-ip}(\cos\theta)$ as $p\to\infty$ ($m=0, 1, 2, \ldots$). We can obtain the required formulas if we use well-known asymptotic formulas (Ref. 7, formulae 3.9.1(1) and 3.9.1(2)) for spherical functions with large complex values of the subscripts, taking into account that

$$P^m_{-1/2-ip}(s) = \mathrm{Re}P^m_{-1/2-ip}(s) = [P^m_{-1/2-ip}(s)+P^m_{-1/2+ip}(s)]/2$$

$$\mathrm{Re}Q^m_{-1/2-ip}(s) = [Q^m_{-1/2-ip}(s)+Q^m_{-1/2+ip}(s)]/2; \quad s=\cos\theta$$

Subsequent application of the asymptotic formulae indicated, as well as the relations

$$\Gamma(1/2+m\pm ip)\Gamma^{-1}(1\pm ip) = (\pm i)^{m-1/2}p^{m-1/2}[1+O(1/p)], \quad p\to\infty$$

which follow from a well-known asymptotic equality (Ref. 7, formula 1.18(4)), to the right-hand sides of these equalities yields the required asymptotic formulae ($m=0, 1, 2, \ldots$)

$$\begin{aligned} P^m_{-1/2-ip}(\cos\theta) &= \operatorname{Re}P^m_{-1/2-ip}(\cos\theta) = \frac{p^{m-1/2}e^{p\theta}}{\sqrt{2\pi\sin\theta}}[1+O(1/p)], \quad p\to\infty \\ \operatorname{Re}Q^m_{-1/2-ip}(\cos\theta) &= \frac{\cos m\pi}{2^{3/2}}\sqrt{\frac{\pi}{\sin\theta}}p^{m-1/2}e^{-p\theta}[1+O(1/p)], \quad p\to\infty \end{aligned} \tag{5.5}$$

Expanding formulae (5.5), it can be shown that the integrals (5.3) converge when $\theta>t$, and we can use contour integration in the plane of the complex variable p to calculate them. As a contour we must take a semicircle with a sufficiently large radius R lying in the upper half-plane $\operatorname{Im}p>0$ with a diameter on the real axis $\operatorname{Im}p=0$. The conditions of Jordan's lemma will be satisfied on the basis of well-known asymptotic formulae (Ref. 7, formulae 3.9.1(1) and 3.9.1(2)), as well as formulae (5.5).

The singular points in the integrands of (5.3) in the upper half-plane will be poles that are among the roots of the denominators of the integrands. The root $p=i/2$ does not lead to a pole in these integrands even when the function $P^2_{\bar\nu}(\cos\omega)$ has a root at $p=i/2$, which can be shown using a well-known identity (Ref. 8, formula 8.753(3)).

This point is also not a singular point for the integrands in (5.3) when $n=1$, despite the fact that the point $p=i/2$ is a root of the function $B_\omega(p, \omega_0)$ by virtue of the same identity.

The next root, $p=3i/2$, is a second-order pole.

The next series of poles are roots of the equation $P^2_{\bar\nu}(\cos\omega)=0$. It can be shown that this equation has a denumerable set of pure imaginary roots $p=iq_j$ ($j=0, 1, 2, \ldots$), which are identical with the roots of the equation

$$P^2_{-1/2+q_j}(\cos\omega) = P^2_{\nu_j}(\cos\omega) = 0, \quad \nu_j = -1/2+q_j$$

The first two roots, $q_0=1/2$ ($\nu_0=0$) and $q_1=3/2$ ($\nu_1=0$), have already been used, and the remaining roots ($\nu_j, j\geq 2$) were found numerically using the MAPLE 6 software package. Their values are listed in Table 1, from which it can be seen that $q_j>3/2$ when $j\geq 2$.

Finally, another series of poles is produced by the roots of the function $B_\omega(p, \omega_0)$ in the upper half-plane $\operatorname{Im}p>0$. As we see from (4.7), the coefficients of the transcendental equation

$$B_\omega(p, \omega_0) = 0 \tag{5.6}$$

are real, and its roots should be conjugate. Therefore, if there is a complex root $p_j=\rho_j+iq^1_j$, then $\bar p_j=\rho_j+iq^1_j$ will also be a root, and the following equality should hold

$$B_\omega(\rho_j+iq^1_j, \omega_0) = B_\omega(\rho_j-iq^1_j, \omega_0)$$

Table 1

ω	$j=2$	3	4	5	6
$\pi/4$	10.28	46.45	50.45	54.46	58.46
$\pi/3$	10.68	19.69	28.71	37.72	46.73
$\pi/6$	9.37	27.78	39.85	51.88	63.90

Table 2

ω	$\frac{G_0}{G_1}$	ω_0	$j=1$	2	3	4	5	6
	0.1	π/4	14.80	22.10	27.02	37.55	47.13	49.73
		π/6	13.32	23.34	30.88	49.35	59.27	85.35
π/4	10	π/8	11.25	21.30	29.27	37.29	45.45	53.67
			9.51	17.54	27.45	33.91	45.95	57.98
	0.1	π/6	10.29	19.12	29.44	77.38	79.15	89.37
		π/6	11.37	20.36	23.31	32.37	35.29	44.37
π/3	10	π/6	9.68	20.28	21.94	32.30	34.02	44.31
		π/4	10.33	18.72	29.17	38.32	41.19	50.33

On the other hand, for spherical functions the equalities

$$P_\nu^m(z) = P_{-\nu-1}^m(z), \quad Q_\nu^m(z) = Q_{-\nu-1}^m(z)$$

hold[7] for any complex values of ν. Taking this into account, we conclude that $\rho_j = 0$, i.e., the roots of Eq. (5.6) are pure imaginary. By virtue of representation (4.9) and the notation $\nu_j^1 = -1/2 + q_j$, transcendental Eq. (5.6) takes on the form

$$\begin{aligned} B_\omega(iq_j, \omega_0) &\equiv gP^2_{\nu_j^1}(\cos\omega_0)[P^2_{\nu_j^1}(\cos\omega_0)Q^2_{\nu_j^1}(\cos\omega) - Q^1_{\nu_j^1}(\cos\omega_0)P_{\nu_j^1}(\cos\omega)] - \\ &- P^1_{\nu_j^1}(\cos\omega_0)[P^2_{\nu_j^1}(\cos\omega_0)Q^2_{\nu_j^1}(\cos\omega) - Q^2_{\nu_j^1}(\cos\omega_0)P^2_{\nu_j^1}(\cos\omega)] = 0, \quad j = 0, 1, 2, \dots \end{aligned}$$

The first root $q_0^1 = -1/2 + \nu_0^1$ is equal to 1/2 by virtue of the identity used (Ref. 8, formula 8.753(3)), and $\nu_0^1 = 0$, which was taken into account above. The remaining roots $\nu_j^1 (j \geq 1)$ were found numerically using the MAPLE 6 software package. Their values are listed in Table 2, from which we see that $q_j^1 > 3/2$, $j \geq 1$.

To calculate the residues at these poles, one must take the following limits.

$$\lim_{\varepsilon\to 0}\frac{\varepsilon}{P^2_{1+\varepsilon}(\cos\omega)} = \left[\sin^2\frac{\omega}{2}\left(3 + \mathrm{tg}^2\frac{\omega}{2}\right)\right]^{-1} = \frac{1}{C_0(\omega)}$$

$$\lim_{\varepsilon\to 0}\frac{d}{d\varepsilon}\frac{\varepsilon}{P^2_{1+\varepsilon}(\cos\omega)} = \frac{3 - \mathrm{tg}^2(\omega/2)[1 + 2\ln(\cos(\omega/2))(2 + \mathrm{cosec}^2(\omega/2))]}{3 + \mathrm{tg}^2(\omega/2)} = C_1(\omega)$$

$$\lim_{\varepsilon\to 0} Q_\omega(i\varepsilon + i3/2, \theta, t) = Q_1^2(\cos\omega)\sin\theta\sin t$$

$$\begin{aligned} &\lim_{\varepsilon\to 0}\frac{d}{d\varepsilon}Q_\omega(i\varepsilon + i3/2, \theta, t) = D_\omega(\theta, t) \\ &\lim_{\varepsilon\to 0}\frac{d}{d\varepsilon}\frac{Q_\omega(i\varepsilon + i3/2, \theta, t)(2\varepsilon + 3)}{(\varepsilon + 3)[1/4 - (\varepsilon + 3/2)^2]} = -\frac{1}{2}D_\omega(\theta, t) + \frac{3}{2}Q_1^2(\cos\omega)\sin\theta\sin t \\ &\frac{d}{d\varepsilon}B_\omega(i\varepsilon + iq_j^1, \omega_0)\Big|_{\varepsilon=0} = \tilde{C}_j(\omega, \omega_0) \\ &\lim_{\varepsilon\to 0} C_\omega(i\varepsilon + i3/2, \omega_0) = C_0(\omega, \omega_0), \quad \lim_{\varepsilon\to 0}\frac{d}{d\varepsilon}C_\omega(i\varepsilon + i3/2, \omega_0) = C_1(\omega, \omega_0) \end{aligned} \tag{5.7}$$

$$D_\omega^0(\theta,t) = [3-C_0(\omega)C_1(\omega)]Q_1^2(\cos\omega)\sin\theta\sin t - D_\omega(\theta,t)$$

$$D_\omega^1(\theta,t) = Q_1^2(\cos\omega)D(\omega,\omega_0)\sin\theta\sin t - D_\omega(\theta,t)$$

$$D(\omega,\omega_0) = [3-C_0(\omega)C_1(\omega)]C_0(\omega,\omega_0) - C_1(\omega,\omega_0)$$

$$D(q) = \frac{2q}{(1/4-q^2)(9/4-q)^2} \tag{5.8}$$

$$E_\omega(q) = \frac{D(q)}{\tilde{P}_\omega(q)}Q_\nu^2(\cos\omega),\quad \tilde{E}_\omega(q) = \frac{D(q)Q_\nu^2(\cos\omega)}{P_\nu^2(\cos\omega)},\quad \tilde{P}_\omega(q) = \frac{dP_\nu^2(\cos\omega)}{d\nu};\quad \nu = -\frac{1}{2}+q$$

$$A_\omega(iq_j^1,\omega_0) = \bar{C}_j(\omega,\omega_0),\quad C_\omega(iq_j,\omega_0) = C_j^*(\omega,\omega_0),\quad \tilde{E}_j(\omega,\omega_0) = \tilde{E}_\omega(q_j^1)\frac{\bar{C}_j(\omega,\omega_0)}{\tilde{C}_j(\omega,\omega_0)}$$

To calculate the limits (5.7), we need the formulae of the derivatives of the spherical functions with respect to the lower index. In our case, the upper index is a positive integer. Using well-known representations (Ref. 7, formulae 3.6.1(2)), we write the formula

$$\frac{dP_\nu^m(\cos\theta)}{d\nu} = \frac{\sin^m\theta}{(-2)^m m!}\sum_{j=0}^{\infty}\frac{(\sin\theta/2)^{2j}}{j!(1+m)_j}\frac{\Gamma(\nu+m+1+j)\Gamma(m-\nu+j)}{\Gamma(\nu-m+1)\Gamma(m-\nu)}\times$$
$$\times[\psi(\nu+m+1+j)-\psi(m-\nu+j)-\psi(\nu-m+1)-\psi(m-\nu)],\quad m = 0,1,2,\ldots \tag{5.9}$$

To calculate $dQ_\nu^m(\cos\theta)/d\nu$, we must use a well-known expansion (Ref. 7, formula 3.5(3)). For example, its use enabled us to obtain the equality

$$Q_1^2(\cos\omega) = 2\operatorname{cosec}^2\omega \tag{5.10}$$

After calculating the residues in the poles enumerated above using formulae (5.9) and (5.10), we find the values of the integrals (5.3)

$$J_n(\xi,\theta,t) = \frac{e^{-3\xi/2}}{2C_0(\omega)}[\xi(\delta_{n0}+\delta_{n1}C_0(\omega,\omega_0))Q_1^2(\cos\omega)\sin\theta\sin t + D_\omega^n(\theta,t)] +$$
$$+\sum_{j=2}^{\infty}E_\omega(q_j)(\delta_{n0}+\delta_{n1}C_j^*(\omega,\omega_0))e^{-q_j\xi}P_{\nu_j}^1(\cos\theta)P_{\nu_j}^1(\cos t) + \tag{5.11}$$
$$+\delta_{n1}\sum_{j=1}^{\infty}\tilde{E}_j(\omega,\omega_0)e^{-q_j^1\xi}\left\{P_{\nu_j^1}^1(\cos\theta)P_{\nu_j^1}^1(\cos t) - \frac{P_{\nu_j^1}^2(\cos\omega)}{Q_{\nu_j^1}^2(\cos\omega)}Q_{\nu_j^1}^1(\cos\theta)P_{\nu_j^1}^1(\cos t)\right\}$$

$\nu_j = -1/2+q_j$ (the roots of the equation $P_{\nu_j}^2(\cos\omega = 0)$) and $\nu_j^1 = -1/2+q_j^1$ (the roots of Eq. (5.6)).

Substituting expressions (5.11) into relations (5.3), (5.2) and (5.1), and then into formula (5.4), we obtain the final form of the solution of the discontinuous boundary-value problem (1.14). The use of this solution along with equality (1.13) enables us to obtain a solution of the auxiliary problem (1.3), (1.5), (1.6), (1.9) in the form

$$u_\varphi(r,\theta) = -a^{3/2}Ar^{-1/2}[F(r,\theta)+\tilde{F}(r,\theta)],\quad a\le r<\infty,\quad 0\le\theta\le\omega \tag{5.12}$$

where

$$\tilde{F}(r,\theta) = g_{01}\sum_{j=1}^{\infty}\left(\frac{3}{2}-q_j^1\right)\tilde{E}_j(\omega,\omega_0)\frac{P^2_{\nu_j^1}(\cos\omega)}{Q^2_{\nu_j^1}(\cos\omega)}\left(\frac{a}{r}\right)^{q_j^1}\begin{cases}Q^1_{\nu_j^1}(\cos\omega_0)P^1_{\nu_j^1}(\cos\theta), & 0\le\theta\le\omega_0\\ P^1_{\nu_j^1}(\cos\omega_0)Q^1_{\nu_j^1}(\cos\theta), & \omega_0\le\theta\le\omega\end{cases}$$

$$F(r,\theta) = \frac{Q_1^2(\cos\omega)}{C_0(\omega)}\left(\frac{A_{\omega_0}}{G_0}+\frac{A_\omega-A_{\omega_0}}{G_1}\right)\left(\frac{a}{r}\right)^{3/2}\sin\theta+S^1_\omega(r,\theta)- \tag{5.13}$$

$$-g_{01}\sum_{j=1}^{\infty}\left(\frac{3}{2}-q_j^1\right)\tilde{E}_j(\omega,\omega_0)P^1_{\nu_j^1}(\cos\omega_0)\left(\frac{a}{r}\right)^{q_j^1}P^1_{\nu_j^1}(\cos\theta),\quad 0\le\theta\le\omega$$

$$S^l_\omega(r,\theta) = \sum_{j=2}^{\infty}\left(\frac{3}{2}-q_j\right)E_\omega(q_j)\left[\frac{I^*_{\omega_0}(\nu_j)}{G_0}+\frac{I^*_\omega(\nu_j)-I^*_{\omega_0}(\nu_j)}{G_1}-\right.$$
$$\left.-g_{01}C_j^*(\omega,\omega_0)P^1_{\nu_j}(\cos\omega_0)\right]\left(\frac{a}{r}\right)^{q_j}P^1_{\nu_j}(\cos\theta),\quad l=1,2;\quad I^*_\omega(\nu)=\int_0^\omega P^1_\nu(\cos t)\sin^2 t\,dt \tag{5.14}$$

6. Solution of the problem of the torsion of a composite cone with a centre of rotation at the apex

According to the presentation of the problem in Section 1, we obtain its solution from the solution (5.12) of the auxiliary problem by taking the limit as $a\to 0$. To do this, we will write the solution (5.12) of the auxiliary problem using equality (1.10) in the form

$$u_\varphi(r,\theta) = -\frac{Mr^{-1/2}}{4\pi A_\omega}\left[\frac{F(r,\theta)}{a^{3/2}}+\frac{\tilde{F}(r,\theta)}{a^{3/2}}\right],\quad a\le r<\infty,\quad 0\le\theta\le\omega \tag{6.1}$$

and take the limit as $a\to 0$ on the right-hand side of the equality obtained.

Taking into account (5.14) and (5.13), as well as the fact that $q_j-3/2>0$ for $j\ge 2$ and $q_j^1-3/2>0$ for $j\ge 1$ (this can be seen from Tables 1 and 2), we arrive at the equalities

$$\lim_{a\to 0}\frac{\tilde{F}(r,\theta)}{a^{3/2}} = 0,\quad \lim_{a\to 0}\frac{F(r,\theta)}{a^{3/2}} = \frac{1}{r^{3/2}}\frac{Q_1^2(\cos\omega)}{C_0(\omega)}\left(\frac{A_{\omega_0}}{G_0}+\frac{A_\omega-A_{\omega_0}}{G_1}\right)\sin\theta \tag{6.2}$$

We take into account that formulae (1.8), (5.7) and (5.10) lead to the equalities

$$\frac{Q_1^2(\cos\omega)}{C_0(\omega)} = \frac{1}{3A_\omega} = \frac{1}{2}\left[\sin^4\frac{\omega}{2}\left(1+2\cos^2\frac{\omega}{2}\right)\right]^{-1} \tag{6.3}$$

By virtue of relations (6.1)–(6.3), the solution of the problem takes the form

$$u_\varphi(r,\theta) = -\frac{M}{12\pi A_\omega}\left(\frac{A_{\omega_0}}{G_0}+\frac{A_\omega-A_{\omega_0}}{G_1}\right)\frac{\sin\theta}{r^2},\quad 0\le r<\infty,\quad 0\le\theta\le\omega \tag{6.4}$$

On the basis of formula (1.4), it hence follows that

$$\tau_{\theta\varphi}(r,\theta)\equiv 0,\quad 0<r<\infty,\quad 0\le\theta\le\omega$$

Thus, for a composite (non-homogeneous) cone with a centre of rotation at its apex, cuts along internal conical surfaces do not cause stress concentration, i.e., the situation revealed for a homogeneous cone.[1] is maintained for a composite cone.

We arrive at the result previously obtained by a fundamentally different method[1] from relation (6.4) by setting $G_0 = G_1 = G$. We have

$$u_\varphi(r, \theta) = -\frac{M}{12\pi G}\frac{\sin\theta}{r^2} = -\frac{M}{4\pi G}\frac{Q_1^2(\cos\omega)}{C_0(\omega)}\frac{\sin\theta}{r^2}, \quad 0 < r < \infty, \quad 0 \le \theta \le \omega \tag{6.5}$$

which is identical with the formula previously obtained in Ref. 1 (where a misprint should be corrected: should be $\gamma_0 = -2A_\omega$ rather than $\gamma_0 = 2A_\omega$).

The situation changes cosiderably if we remove the apex from the composite cone and apply the stresses (1.7) or the equivalent torque (1.8) on the spherical surface $r = a$. In the latter case, the quantity A in relation (1.7) must be represented in the form

$$A = M[4\pi A_\omega a^3]^{-1} \tag{6.6}$$

The displacement $u_\varphi(r, \theta)$ for such a cone will be represented by (5.12), in which A can be replaced by the right-hand side of equality (6.6). Using formula (1.4) and taking (1.1) into account, from these displacements we find the stress

$$\tau_{\theta\varphi}(r, \theta) = -A\begin{Bmatrix} G_0, & 0 \le \theta \le \omega_0 \\ G_1, & \omega_0 < \theta \le \omega \end{Bmatrix}[S_\omega^2(r, \theta) + \tilde{F}^{\bullet}(r, \theta)]\left(\frac{a}{r}\right)^{3/2}, \quad a \le r < \infty, \quad 0 \le \theta \le \omega \tag{6.7}$$

The function S_ω^2 is defined by (5.15).

As we see, here $\tau_{\theta\varphi}(r, \theta) \ne 0$. Therefore, if the apex is removed from a composite cone and the stresses (1.7), which are equivalent to the torque (1.8), are applied instead of it, stress concentration will occur in the general case along cuts on the internal conical surfaces.

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